

REARRANGEMENTS AND RADIAL GRAPHS OF CONSTANT MEAN CURVATURE IN HYPERBOLIC SPACE

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ABSTRACT. We investigate the problem of finding smooth hypersurfaces of constant mean curvature in hyperbolic space, which can be represented as radial graphs over a subdomain of the upper hemisphere. Our approach is variational and our main results are proved via rearrangement techniques.

1. INTRODUCTION

In this paper we study the problem of finding smooth hypersurfaces of constant mean curvature in hyperbolic space \mathbb{H}^{n+1} , which can be represented as radial graphs over a domain Ω strictly contained in the upper hemisphere $\mathbb{S}_+^n \subset \mathbb{R}^{n+1}$. This also leads by an approximation process to the existence and uniqueness of smooth complete hypersurfaces of constant mean curvature $H \in (-1, 1)$ with prescribed asymptotic boundary Γ at infinity, in case Γ is the boundary of a continuous star-shaped domain.

We use the half-space model,

$$\mathbb{H}^{n+1} = \{(x, x_{n+1}) \in \mathbb{R}^{n+1} | x_{n+1} > 0\}$$

equipped with the hyperbolic metric

$$ds_H^2 = \frac{1}{x_{n+1}^2} ds_E^2,$$

where ds_E^2 denotes the Euclidean metric on \mathbb{R}^{n+1} .

Let $\Omega \subset \mathbb{S}_+^n$, and suppose that Σ is a radial graph over Ω with position vector X in \mathbb{R}^{n+1} . Then we can write

$$(1.1) \quad X = e^{v(z)} z, \quad z \in \Omega,$$

for a function v defined over Ω . Assume that Σ has constant mean curvature H in hyperbolic space with respect to the outward unit normal. Then v satisfies the divergence form elliptic equation

$$(1.2) \quad \operatorname{div}_z \left(\frac{y^{-n} \nabla v}{\sqrt{1 + |\nabla v|^2}} \right) = nH y^{-(n+1)} \quad \text{in } \Omega,$$

where $y = z_{n+1}$ and the divergence and gradient are with respect to the standard metric on the sphere.

We apply direct methods of the calculus of variations, in order to prove the existence of a smooth solution to (1.2). Let,

$$(1.3) \quad \mathcal{I}_\Omega(v) := \int_\Omega \sqrt{1 + |\nabla v|^2} y^{-n} dz + nH \int_\Omega v(z) y^{-(n+1)} dz,$$

be the energy functional associated to equation (1.2). In this variational setting, we will easily obtain the existence of bounded local minimizers of $\mathcal{I}_\Omega(\cdot)$ in the class $BV(\Omega)$, as long as $|H| < 1$. However the Dirichlet problem in this generality needs to be carefully formulated, see Section 2.

Our main objective is to prove a regularity result which guarantees that such minimizers are smooth, and hence the associated graphs (1.1) are smooth hyper-surfaces of constant mean curvature in \mathbb{H}^{n+1} . We first prove the following result.

Theorem 1.1. *Assume $n \leq 6$ and let $v \in BV(\Omega) \cap L^\infty(\Omega)$ be a local minimizer to $\mathcal{I}_\Omega(\cdot)$. Then $v \in C^\infty(\Omega)$.*

The elegance of this low dimensional result lies in the fact that it does not require any kind of a priori gradient bounds, which in this context may appear computationally tedious. The proof is based on the connection between non-parametric (radial graphs) and parametric surfaces of constant mean curvature in hyperbolic space. For the latter, regularity in low dimensions is well-known (see for example [6]). We exploit this fact and recover the same regularity result for radial graphs, via rearrangement techniques. A similar approach has been followed in the Euclidean setting to find smooth vertical graphs of prescribed mean curvature (see for example [3]). See also [1] for an existence and regularity result for a degenerate equation obtained via similar techniques.

In order to remove the low dimensional constraint we first analyze the case when the domain Ω satisfies an appropriate assumption. This allows us to set up and solve the Dirichlet problem for $\mathcal{I}_\Omega(\cdot)$ and obtain smoothness of the minimizer from the smoothness of the boundary data. Indeed, we prove the following result which requires the construction of appropriate barriers.

Theorem 1.2. *Let Ω be a subdomain of \mathbb{S}_+^n with $\partial\Omega \in C^2$, and let γ be a continuous radial graph over $\partial\Omega$. Let h be the hyperbolic mean curvature of the radial cone over $\partial\Omega$ restricted to $\partial\Omega$. Then if $h > |H|$, there exists a unique smooth radial graph Σ of constant mean curvature H in \mathbb{H}^{n+1} (defined over Ω) with boundary γ .*

Then using standard approximation techniques, a corollary of Theorem 1.2, and an interior gradient bound which is of independent interest, we prove the following result.

Theorem 1.3. *Let $v \in BV(\Omega) \cap L^\infty(\Omega)$ be a local minimizer to $\mathcal{I}_\Omega(\cdot)$. Then $v \in C^\infty(\Omega)$.*

Finally, by a limiting argument using the afore mentioned barriers we recover the following result from [4].

Theorem 1.4. *Let Γ be the boundary of a continuous star-shaped domain in \mathbb{R}^n and let $|H| < 1$. Then there exists a unique hypersurface Σ of constant mean curvature H in \mathbb{H}^{n+1} with asymptotic boundary Γ . Moreover Σ may be represented as the radial graph over $\mathbb{S}_+^n \subset \mathbb{R}^{n+1}$ of a function in $C^\infty(\mathbb{S}_+^n) \cap C^0(\overline{\mathbb{S}_+^n})$.*

The paper is organized as follows. In Section 2, after briefly introducing some notation, we set up and solve the Dirichlet problem for our energy functional in the class of BV functions. Then, in Section 3, we prove our low dimensional result Theorem 1.1, via rearrangement techniques. The proof of Theorem 1.2 is exhibited in Section 4. In Section 5, we present the proof of an interior gradient bound for smooth solutions to equation (1.2) and then we apply it together with a Corollary of Theorem 1.2 to remove the dimensional constraint and prove Theorem 1.3. Finally, we conclude Section 5 by sketching the proof of Theorem 1.4.

2. THE DIRICHLET PROBLEM FOR THE ENERGY FUNCTIONAL $\mathcal{I}_\Omega(\cdot)$

2.1. Notation. Throughout this paper we denote by \mathbb{S}^n the standard unit sphere in \mathbb{R}^{n+1} and by \mathbb{S}_+^n the upper hemisphere. We use div_z and ∇ to denote respectively the divergence and the covariant gradient on \mathbb{S}^n . Also, we let \mathbf{e} be the unit vector in the positive x_{n+1} direction in \mathbb{R}^{n+1} and

$$y = \mathbf{e} \cdot z, \quad \text{for } z \in \mathbb{S}^n,$$

where ‘ \cdot ’ denotes the Euclidean inner product in \mathbb{R}^{n+1} .

We recall the following fact, which will be used in the proof of the existence of minimizers in the next subsection.

Remark 2.1. Assume $|H| < 1$. Let $B_R(a)$ be a ball of radius R centered at $a = (a', -HR) \in \mathbb{R}^{n+1}$ where $a' \in \mathbb{R}^n$. Then $S = \partial B_R(a) \cap \mathbb{H}^{n+1}$ has constant hyperbolic mean curvature H with respect to its outward normal. Analogously, let $B_R(b)$ be a ball of radius R centered at $b = (b', HR) \in \mathbb{R}^{n+1}$ where $b' \in \mathbb{R}^n$. Then $S = \partial B_R(b) \cap \mathbb{H}^{n+1}$ has constant hyperbolic mean curvature H with respect to its inward normal.

2.2. Existence of minimizers. We now formulate and solve the Dirichlet problem for the functional $\mathcal{I}_\Omega(\cdot)$ in the Introduction.

Let $\Omega \subset \mathbb{S}_+^n$; for a function $v \in BV(\Omega)$ define,

$$\begin{aligned} \int_\Omega \sqrt{1 + |\nabla v|^2} y^{-n} &:= \sup \left\{ \int_\Omega v(z) \operatorname{div}_z [\tilde{\gamma} y^{-n}] dz + \int_\Omega \gamma_{n+1} y^{-n} dz : \right. \\ &\quad \left. \gamma = (\tilde{\gamma}, \gamma_{n+1}) \in C_0^1(\Omega, T\Omega \times \mathbb{R}), |\tilde{\gamma}|_{\mathbb{S}^n}^2 + |\gamma_{n+1}|^2 \leq 1 \right\}. \end{aligned}$$

Here we are denoting with dz and $|\cdot|_{\mathbb{S}^n}$ respectively the measure and the length on the standard unit sphere.

Let $v \in BV(\Omega)$ and define the energy functional

$$\mathcal{I}_\Omega(v) := \int_\Omega \sqrt{1 + |\nabla v|^2} y^{-n} + nH \int_\Omega v(z) y^{-(n+1)} dz,$$

where H is a constant with $|H| < 1$. In what follows we denote

$$\begin{aligned} A_\Omega(v) &:= \int_\Omega \sqrt{1 + |\nabla v|^2} y^{-n}, \\ V_\Omega(v) &:= \int_\Omega v(z) y^{-(n+1)} dz. \end{aligned}$$

We omit the subscript Ω from the definitions above, whenever there is no possibility of confusion.

The Dirichlet problem for the energy functional $\mathcal{I}_\Omega(\cdot)$ consists in minimizing this functional among all $v \in BV(\Omega)$ whose trace on $\partial\Omega$ is a prescribed function $\phi \in L^1(\partial\Omega)$. However, this problem may not be solvable in such generality. The following proposition suggests an alternative form of the Dirichlet problem.

Proposition 2.2. *Assume $\partial\Omega$ is C^1 and let $\phi \in L^1(\partial\Omega)$. Then,*

$$\inf\{\mathcal{I}(v) : v \in BV(\Omega), v = \phi \text{ on } \partial\Omega\} = \inf\{\mathcal{I}(v) + \int_{\partial\Omega} |v - \phi| y^{-n} dH_{n-1}; v \in BV(\Omega)\}.$$

Proof. Let $v \in BV(\Omega)$ and let $\epsilon > 0$. Gagliardo's Theorem (see Theorem 2.16 of [3]) states that there exists a function $w \in W^{1,1}(\Omega)$ with $w = v - \phi$ on $\partial\Omega$ and

$$(2.1) \quad \int_{\Omega} |\nabla w| y^{-n} \leq (1 + \epsilon) \int_{\partial\Omega} |v - \phi| y^{-n} dH_{n-1},$$

$$(2.2) \quad n|H| \int_{\Omega} |w| y^{-(n+1)} \leq \epsilon \int_{\partial\Omega} |v - \phi| y^{-n} dH_{n-1}.$$

The function $u = v + w$ is in $BV(\Omega)$ and $u = \phi$ on $\partial\Omega$. Moreover, by (2.1)

$$\begin{aligned} \int_{\Omega} \sqrt{1 + |\nabla u|^2} y^{-n} &\leq \int_{\Omega} \sqrt{1 + |\nabla v|^2} y^{-n} + \int_{\Omega} |\nabla w| y^{-n} \\ &\leq \int_{\Omega} \sqrt{1 + |\nabla v|^2} y^{-n} + (1 + \epsilon) \int_{\partial\Omega} |v - \phi| y^{-n} dH_{n-1}. \end{aligned}$$

Thus, by (2.2)

$$\mathcal{I}(u) \leq \mathcal{I}(v) + (1 + 2\epsilon) \int_{\partial\Omega} |v - \phi| y^{-n} dH_{n-1}.$$

As ϵ tends to zero, taking the infimum over all $v \in BV(\Omega)$ we obtain

$$\begin{aligned} \inf\{\mathcal{I}(v) : v \in BV(\Omega), v = \phi \text{ on } \partial\Omega\} &\leq \\ \inf\{\mathcal{I}(v) + \int_{\partial\Omega} |v - \phi| y^{-n} dH_{n-1}; v \in BV(\Omega)\}, \end{aligned}$$

which suffices as the opposite inequality is trivial. \square

Proposition 2.2 suggests the introduction of the modified energy functional

$$\mathcal{I}_\Omega^\phi(v) = \mathcal{I}(v) + \int_{\partial\Omega} |v - \phi| y^{-n} dH_{n-1}.$$

Again the dependence on Ω will be made explicit only when strictly necessary.

A compactness argument allows us to conclude that the minimization problem for $\mathcal{I}^\phi(\cdot)$ is always solvable in the appropriate class of functions. Precisely we have the following Theorem.

Theorem 2.3. *Assume $\partial\Omega$ is Lipschitz continuous and let $\phi \in L^\infty(\partial\Omega)$. Then, $\mathcal{I}^\phi(\cdot)$ attains its minimum u in $BV(\Omega)$. Moreover $u \in L^\infty(\Omega)$ and $\|u\|_{L^\infty} \leq M$ for some $M = M(\|\phi\|_\infty)$.*

Proof. Let $S_\delta := \{y > \delta\} \cap \mathbb{S}_+^n$ contain $\overline{\Omega}$ and let us extend ϕ to a $W^{1,1}$ function in $S_\delta \setminus \overline{\Omega}$ that we will still denote by ϕ . Let $v \in BV(\Omega)$ and define

$$v_\phi = \begin{cases} v(z), & z \in \Omega; \\ \phi, & z \in S_\delta \setminus \Omega. \end{cases}$$

Then, $v_\phi \in BV(S_\delta)$ and by the trace formula

$$\begin{aligned} \int_{S_\delta} \sqrt{1 + |\nabla v_\phi|^2} y^{-n} &= \int_{\Omega} \sqrt{1 + |\nabla v|^2} y^{-n} + \int_{S_\delta \setminus \Omega} \sqrt{1 + |\nabla \phi|^2} y^{-n} \\ &\quad + \int_{\partial\Omega} |v - \phi| y^{-n} dH_{n-1}. \end{aligned}$$

Therefore,

$$\mathcal{I}_{S_\delta}(v_\phi) = \mathcal{I}_\Omega^\phi(v) + C(\phi),$$

where $C(\phi)$ is a constant independent of v . Hence in order to minimize $\mathcal{I}_\Omega^\phi(\cdot)$ among all $BV(\Omega)$ functions, it suffices to minimize $\mathcal{I}_{S_\delta}(\cdot)$ among all functions $u \in BV(S_\delta)$, coinciding with ϕ in $S_\delta \setminus \overline{\Omega}$.

Let $\overline{\varphi}$ and $\underline{\varphi}$ be smooth solutions to the equation

$$(2.3) \quad \operatorname{div}_{\mathbb{S}^n} \left(\frac{y^{-n} \nabla v}{\sqrt{1 + |\nabla v|^2}} \right) = nH y^{-(n+1)}, \quad \text{in } S_\delta$$

such that

$$(2.4) \quad \inf_{S_\delta} \overline{\varphi} > \|\phi\|_{L^\infty(S_\delta)},$$

and

$$(2.5) \quad \sup_{S_\delta} \underline{\varphi} < -\|\phi\|_{L^\infty(S_\delta)}.$$

The existence of $\overline{\varphi}$ and $\underline{\varphi}$ follows from Remark 2.1 by choosing $a' = 0$ for a suitable choice of R . Explicitly,

$$\underline{\varphi} = -\|\phi\|_{L^\infty(S_\delta)} + \log(\sqrt{H^2 y^2 + (1 - H^2)} - Hy),$$

$$\overline{\varphi} = \underline{\varphi} + 2\|\phi\|_{L^\infty(S_\delta)} - \log(1 - H).$$

Now, let $u_j \in BV(S_\delta)$ be a minimizing sequence, that is

$$\inf\{\mathcal{I}_{S_\delta}(u) : u \in BV(S_\delta), u = \phi \text{ in } S_\delta \setminus \overline{\Omega}\} = \lim_j \mathcal{I}_{S_\delta}(u_j) = I.$$

Let us approximate the u_j 's with smooth functions which we still denote by u_j 's. Set

$$\overline{u}_j = \min\{u_j, \overline{\varphi}\}$$

and compute

(2.6)

$$\begin{aligned} \mathcal{I}_{S_\delta}(u_j) &= \mathcal{I}_{S_\delta \cap \{u_j < \overline{\varphi}\}}(u_j) + \mathcal{I}_{S_\delta \cap \{u_j > \overline{\varphi}\}}(u_j) \\ &= \mathcal{I}_{S_\delta}(\overline{u}_j) - \mathcal{I}_{S_\delta \cap \{u_j > \overline{\varphi}\}}(\overline{u}_j) + \mathcal{I}_{S_\delta \cap \{u_j > \overline{\varphi}\}}(u_j) \\ &= \mathcal{I}_{S_\delta}(\overline{u}_j) + \int_{S_\delta \cap \{u_j > \overline{\varphi}\}} \left[y^{-n} \left(\sqrt{1 + |\nabla u_j|^2} - \sqrt{1 + |\nabla \overline{u}_j|^2} \right) \right. \\ &\quad \left. + nH(u_j - \overline{u}_j)y^{-(n+1)} \right] dz \\ (2.7) \quad &\geq \mathcal{I}_{S_\delta}(\overline{u}_j) + \int_{S_\delta \cap \{u_j > \overline{\varphi}\}} \left(\frac{y^{-n} \nabla \overline{u}_j}{\sqrt{1 + |\nabla \overline{u}_j|^2}} \nabla(u_j - \overline{u}_j) \right. \\ &\quad \left. + nH(u_j - \overline{u}_j)y^{-(n+1)} \right) dz. \end{aligned}$$

After integration by parts the integral in (2.7) is identically zero in view of the fact that $\overline{\varphi}$ satisfies (2.3)-(2.4). Hence,

$$(2.8) \quad \mathcal{I}_{S_\delta}(u_j) \geq \mathcal{I}_{S_\delta}(\overline{u}_j).$$

Analogously, set

$$\underline{u}_j = \max\{\underline{\varphi}, \overline{u}_j\}$$

and note that

$$\underline{\varphi} \leq \underline{u}_j \leq \overline{\varphi}.$$

Then,

$$\begin{aligned}
 \mathcal{I}_{S_\delta}(\underline{u}_j) &= \mathcal{I}_{S_\delta \cap \{\bar{u}_j < \underline{\varphi}\}}(\underline{u}_j) + \mathcal{I}_{S_\delta \cap \{\bar{u}_j > \underline{\varphi}\}}(\underline{u}_j) \\
 &= \mathcal{I}_{S_\delta}(\bar{u}_j) - \mathcal{I}_{S_\delta \cap \{\bar{u}_j < \underline{\varphi}\}}(\bar{u}_j) + \mathcal{I}_{S_\delta \cap \{\bar{u}_j < \underline{\varphi}\}}(\underline{\varphi}) \\
 &= \mathcal{I}_{S_\delta}(\bar{u}_j) + \int_{S_\delta \cap \{\bar{u}_j < \underline{\varphi}\}} \left[y^{-n} \left(\sqrt{1 + |\nabla \underline{\varphi}|^2} - \sqrt{1 + |\nabla \bar{u}_j|^2} \right) \right. \\
 &\quad \left. + nH(\underline{\varphi} - u_j)y^{-(n+1)} \right] dz \\
 &\leq \mathcal{I}_{S_\delta}(\bar{u}_j) + \int_{S_\delta \cap \{\bar{u}_j < \underline{\varphi}\}} \left(\frac{y^{-n} \nabla \underline{\varphi}}{\sqrt{1 + |\nabla \underline{\varphi}|^2}} \nabla(\underline{\varphi} - u_j) \right. \\
 &\quad \left. + nH(\underline{\varphi} - u_j)y^{-(n+1)} \right) dz.
 \end{aligned}$$

Again since $\underline{\varphi}$ satisfies (2.3)-(2.5), the last term vanishes. Hence,

$$(2.9) \quad \mathcal{I}_{S_\delta}(\bar{u}_j) \geq \mathcal{I}_{S_\delta}(\underline{u}_j).$$

Combining (2.8) and (2.9) we obtain that

$$\lim_j \mathcal{I}_{S_\delta}(u_j) \geq K,$$

for some constant K , hence I is finite. Moreover, the \underline{u}_j 's are uniformly bounded in BV (since $\mathcal{I}_{S_\delta}(\underline{u}_j) \leq \mathcal{I}_{S_\delta}(u_j) \leq C$, as the u_j 's are a minimizing sequence) and we can extract a subsequence which converges in $L^1(S_\delta)$ to some function $u \in BV(S_\delta)$. Furthermore $u \in L^\infty(\Omega)$ and $u = \phi$ in $S_\delta \setminus \bar{\Omega}$. Then by the lower semicontinuity of our functional we find that u is the required minimizer. \square

We now collect a few more facts about minimizers, which will be used in the next sections.

Remark 2.4. From the strict convexity of our functional, in particular

$$\frac{\mathcal{I}^\phi(v_1) + \mathcal{I}^\phi(v_2)}{2} \geq \mathcal{I}^\phi\left(\frac{v_1 + v_2}{2}\right),$$

we obtain that if v_1, v_2 are two minima of $\mathcal{I}^\phi(\cdot)$, then $v_1 = v_2 + \text{const}$. Moreover on $\partial\Omega$ the traces of v_1 and v_2 satisfy $(v_1 - \phi)(v_2 - \phi) > 0$. Finally, if v_1 and v_2 have the same trace ϕ on $\partial\Omega$ then $v_1 = v_2$.

Corollary 2.5. *Let v minimize $\mathcal{I}^\phi(\cdot)$, and $\phi \in C(\partial\Omega)$. Assume that $\bar{\varphi}$ (resp. $\underline{\varphi}$) is a smooth supersolution (resp. subsolution) to equation (1.2), with $\bar{\varphi} \geq \phi$ (resp. $\underline{\varphi} \leq \phi$) on $\partial\Omega$. Then $\bar{\varphi} \geq v$ (resp. $\underline{\varphi} \leq v$) in Ω .*

The corollary above follows by the same argument as in the proof of Theorem 2.3 (in particular see formula (2.6)), together with Remark 2.4.

Lemma 2.6. *Let $v_i \in BV(\Omega)$ minimize \mathcal{I}^{ϕ_i} , $\phi_i \in L^\infty(\partial\Omega)$, $v_i = \phi_i$ on $\partial\Omega$ (in the trace sense) $i = 1, 2$. Assume $\phi_1 \geq \phi_2$ on $\partial\Omega$. Then $v_1 \geq v_2$ in Ω .*

Proof. Set

$$v_{max} = \max\{v_1, v_2\}, \quad v_{min} = \min\{v_1, v_2\}.$$

Then,

$$(2.10) \quad \mathcal{I}(v_1) + \mathcal{I}(v_2) \geq \mathcal{I}(v_{max}) + \mathcal{I}(v_{min}).$$

Indeed formula (2.10) clearly holds in the case when v_1 and v_2 are smooth. We can then approximate v_i , $i = 1, 2$ with a sequence $\{v_i^m\}$ of smooth functions such that $v_i^m \rightarrow v_i$ in L^1 and $A(v_i^m) \rightarrow A(v_i)$. Then by the lower semicontinuity of our functional we immediately get (2.10) for BV functions.

Moreover, since $\phi_1 \geq \phi_2$ on $\partial\Omega$, we have that v_{max} has the same trace as v_1 while v_{min} has the same trace as v_2 on $\partial\Omega$. The desired claim now follows by the uniqueness of minimizers (Remark 2.4). \square

Remark 2.7. It is straightforward to show that smooth solutions to the Dirichlet problem for the divergence equation (1.2) on Ω and boundary data ϕ , also minimize the energy integral $\mathcal{I}(\cdot)$ among all competitors equal to ϕ on $\partial\Omega$.

3. REGULARITY IN LOW DIMENSIONS

In this section we prove our main regularity result Theorem 1.1. The existence of local bounded minimizers is guaranteed by Theorem 2.3.

We proceed to investigate the connection between non-parametric and parametric surfaces of constant mean curvature in hyperbolic space.

For any function v over Ω we set

$$V := \{x \in \mathbb{R}^{n+1} : x = e^w z, z \in \Omega, -\infty < w < v(z)\}.$$

V is the subgraph of the radial graph defined by

$$X = e^{v(z)} z, \quad z \in \Omega.$$

Also, for any $T > 0$, we define

$$C_T := \{x \in \mathbb{R}^{n+1} : x = e^w z, z \in \Omega, -T - 1 < w < T + 1\},$$

$$\underline{C}_T := \{x \in \mathbb{R}^{n+1} : x = e^w z, z \in \Omega, -T - 1 < w < -T\},$$

$$\overline{C}_T := \{x \in \mathbb{R}^{n+1} : x = e^w z, z \in \Omega, -T - 1 < w < T\}.$$

Let us denote by

$$\mathcal{E} := \{E \subseteq C : E \text{ measurable}, \underline{C}_T \subseteq E \subseteq \overline{C}_T\}.$$

Also, let us define the set functionals representing respectively the perimeter and the volume in C_T of a set U in the hyperbolic space \mathbb{H}^{n+1} :

$$\mathcal{P}_{C_T}(U) := \sup \left\{ \int_{C_T} \varphi_U \operatorname{div}_x (g x_{n+1}^{-n}) dx : g \in C_0^1(C_T; \mathbb{R}^{n+1}), |g|^2 \leq 1 \right\}$$

$$\operatorname{Vol}(U) := \int_{U \cap C_T} x_{n+1}^{-(n+1)} dx.$$

Here we denote by φ_U the characteristic function of a set U . We will often drop the subscript C_T , whenever this generates no confusion.

Set,

$$\mathcal{F}(U) = \mathcal{P}(U) + nH \operatorname{Vol}(U).$$

We wish to prove the following theorem.

Theorem 3.1. *Let $v \in BV(\Omega) \cap L^\infty(\Omega)$ be a local minimizer of $\mathcal{I}(\cdot)$ and let $T > \|v\|_{L^\infty}$. Then $V_T := V \cap C_T$ locally minimizes $\mathcal{F}(\cdot)$ among all competitors in \mathcal{E} .*

We start with the following proposition.

Proposition 3.2. *Let $v \in BV(\Omega) \cap L^\infty(\Omega)$, and let $T > \|v\|_{L^\infty}$. Then,*

$$(3.1) \quad \mathcal{F}(V_T) = \mathcal{I}(v) + k(T + 1)$$

where $V_T := V \cap C_T$ and $k = \int_{\Omega} y^{-(n+1)} dz$.

Proof. We start by showing that

$$\mathcal{P}(V_T) \geq A(v).$$

By definition, for any g compactly supported in C_T satisfying $|g|^2 \leq 1$, we have

$$(3.2) \quad \mathcal{P}(V_T) \geq \int_{V_T} \operatorname{div}_x [x_{n+1}^{-n} g(x)] dx =$$

$$(3.3) \quad = \int_{\Omega} \int_{-T-1}^{v(z)} \operatorname{div}_{z,w} [y^{-n} g(z, w)] dw dz$$

where in the second line we performed the change of variable $x = e^w z$. Also we denote by $\operatorname{div}_{z,w}$ the divergence on the manifold $\mathbb{S}^n \times \mathbb{R}$ with the standard product metric. Notice that $g(z, w) = (\tilde{g}(z, w), g_{n+1}(z, w))$ satisfies $|\tilde{g}|_{\mathbb{S}^n}^2 + |g_{n+1}|^2 \leq 1$. Since g is arbitrary, we can choose

$$(\tilde{g}(z, w), g_{n+1}(z, w)) = (\tilde{\gamma}(z), \gamma_{n+1}(z)) \eta(w),$$

where $\gamma = (\tilde{\gamma}, \gamma_{n+1})$ is a vector field compactly supported on Ω such that $|\tilde{\gamma}|_{\mathbb{S}^n}^2 + |\gamma_{n+1}|^2 \leq 1$, while η is compactly supported in $[-T-1, \sup_{\Omega} v + 1]$ and such that $\eta \equiv 1$ on $[-T, \sup_{\Omega} v]$ and $|\eta| \leq 1$. Thus,

$$\begin{aligned} \mathcal{P}(V_T) &\geq \int_{\Omega} \int_{-T-1}^{v(z)} \operatorname{div}_{z,w} [\gamma \eta y^{-n}] dw dz = \\ &\int_{\Omega} \int_{-T-1}^{v(z)} \operatorname{div}_z [\tilde{\gamma} y^{-n}] \eta(w) dw dz + \int_{\Omega} \int_{-T-1}^{v(z)} \gamma_{n+1}(z) y^{-n} \eta'(w) dw dz. \end{aligned}$$

From our choice of η we have that

$$\int_{-T-1}^{v(z)} \eta'(w) dw = 1$$

and

$$\int_{-T-1}^{v(z)} \eta(w) dw = v(z) - c$$

with c constant.

Thus,

$$\mathcal{P}(V_T) \geq \int_{\Omega} v(z) \operatorname{div}_z [\tilde{\gamma} y^{-n}] dz + \int_{\Omega} \gamma_{n+1} y^{-n} dz,$$

and the desired statement follows by taking the sup over all $\gamma = (\tilde{\gamma}, \gamma_{n+1})$ of length smaller than 1, compactly supported in Ω .

The opposite inequality follows by a standard limiting argument. In the case when $v \in C^1(\Omega)$ then clearly

$$\mathcal{P}(V_T) = A(v).$$

Now let $v_j \in C^\infty(\Omega)$, $v_j \rightarrow v$ in $L^1(\Omega)$ and $A(v_j) \rightarrow A(v)$. Then $V_{j,T} \rightarrow V_T$ in $L^1(C)$ and therefore by the lower semicontinuity of the perimeter functional we get

$$\mathcal{P}(V_T) \leq \liminf_{j \rightarrow \infty} \mathcal{P}(V_{j,T}) = \lim_{j \rightarrow \infty} A(v_j) = A(v).$$

Finally, we compute

$$\operatorname{Vol}(V_T) = \int_{V_T} x_{n+1}^{-(n+1)} dx = \int_{\Omega} \int_{-T-1}^{v(z)} y^{-(n+1)} dw dz = V(v) + k(T+1),$$

which concludes the proof. \square

Let $E \in \mathcal{E}$ and denote by \tilde{E} the image of E under the coordinate transformation $x = e^w z$, $z \in \Omega$, $-T-1 < w < T+1$. Set

$$(3.4) \quad u(z) = \int_{-T}^T \varphi_{\tilde{E}}(z, w) dw - T, \quad z \in \Omega.$$

The subgraph in C_T of the radial surface $X = e^{u(z)} z$, $z \in \Omega$ is the rearrangement of the set E in the radial direction.

Proposition 3.3. *For any $E \in \mathcal{E}$ we have,*

$$(3.5) \quad \mathcal{F}(E) \geq \mathcal{I}(u) + k(T+1),$$

where $k = \int_{\Omega} y^{-(n+1)} dz$.

Proof. According to the definition,

$$\mathcal{P}(E) \geq \int_C \varphi_E \operatorname{div}_x [x_{n+1}^{-n} g(x)] dx = \int_{\Omega} \int_{-T-1}^T \varphi_{\tilde{E}} \operatorname{div}_{z,w} [y^{-n} g(z, w)] dw dz$$

after performing the change of variable $x = e^w z$. As in Proposition 3.2 since g is arbitrary, we can choose

$$(\tilde{g}(z, w), g_{n+1}(z, w)) = (\tilde{\gamma}(z), \gamma_{n+1}(z)) \eta(w),$$

where $\gamma = (\tilde{\gamma}, \gamma_{n+1})$ is a vector field compactly supported on Ω such that $|\tilde{\gamma}|_{\mathbb{S}^n}^2 + |\gamma_{n+1}|^2 \leq 1$, while η is compactly supported in $[-T-1, T+1]$ and such that $\eta \equiv 1$ on $[-T, T]$ and $|\eta| \leq 1$. Thus,

$$\begin{aligned} \mathcal{P}(E) &\geq \int_{\Omega} \int_{-T-1}^T \varphi_{\tilde{E}} \operatorname{div}_{z,w} [\gamma \eta y^{-n}] dw dz = \\ &\int_{\Omega} \int_{-T-1}^T \varphi_{\tilde{E}} \operatorname{div}_z [\tilde{\gamma} y^{-n}] \eta(w) dw dz + \int_{\Omega} \int_{-T-1}^T \varphi_{\tilde{E}} \gamma_{n+1}(z) y^{-n} \eta'(w) dw dz. \end{aligned}$$

From our choice of η we have that

$$\int_{-T-1}^{-T} \eta'(w) dw = 1,$$

and also $\varphi_{\tilde{E}}(z, w) \equiv 1$ for $-T-1 < w < -T$. Thus, according to the definition of u we have

$$\mathcal{P}(E) \geq \int_{\Omega} u(z) \operatorname{div}_z [\tilde{\gamma} y^{-n}] dz + \int_{\Omega} \gamma_{n+1} y^{-n} dz,$$

and the desired statement follows by taking the sup over all $\gamma = (\tilde{\gamma}, \gamma_{n+1})$ of length smaller than 1, compactly supported in Ω .

Finally, we compute

$$(3.6) \quad \begin{aligned} \operatorname{Vol}(E) &= \int_C \varphi_E x_{n+1}^{-(n+1)} dx = \int_{\Omega} (u(z) + T) y^{-(n+1)} dz + \\ &\int_{\Omega} \int_{-T-1}^{-T} y^{-(n+1)} dw dz = V(u) + k(T+1), \end{aligned}$$

which concludes the proof. \square

We are now ready to prove our Theorem.

Proof of Theorem 3.1. Let $A \subset\subset \Omega$ and let $E \in \mathcal{E}$ coincide with V_T outside a compact set in $\{x \in \mathbb{R}^{n+1} : x = e^w z, z \in A, -T-1 < w < T+1\}$. Then the function u associated to E coincides with v outside of A and hence according to (3.1) and (3.5),

$$\mathcal{F}(V_T) \leq \mathcal{I}(v) + k(T+1) \leq \mathcal{I}(u) + k(T+1) \leq \mathcal{F}(E).$$

□

Since V_T locally minimizes \mathcal{F} in \mathcal{E} , it is known that the boundary of V_T is a regular (analytic) hypersurface outside a closed set S , with $H_{n-6}(S) = 0$ (see [6]). As an immediate corollary we shall prove that v is regular in $L = \Omega \setminus \text{proj}_\Omega S$.

Towards this aim, we need to recall the following lemma that can be found in [4].

Lemma 3.4. *Let Σ be a constant mean curvature hypersurface in \mathbb{H}^{n+1} with position vector X in \mathbb{R}^{n+1} and unit normal ν with respect to the Euclidean metric. Let $|A|$ and Δ denote respectively the norm of the second fundamental form of Σ and the Laplace-Beltrami operator on Σ with respect to the hyperbolic metric. Then,*

$$(3.7) \quad \Delta \frac{X \cdot \nu}{u} = (n - |A|^2) \frac{X \cdot \nu}{u},$$

where u denotes the height function $u = X \cdot \mathbf{e}$.

Corollary 3.5. *Let $v \in BV(\Omega) \cap L^\infty(\Omega)$ be a local minimizer to $\mathcal{I}(\cdot)$. Then $v \in C^\infty(L)$ with $H_{n-6}(\Omega \setminus L) = 0$.*

Proof. Let Σ be the radial graph associated to v . We use the notation from Lemma 3.4. Assume by contradiction that $X \cdot \nu = 0$ at some point $z \in L$. Then,

$$X \cdot \nu \geq 0.$$

Hence according to (3.7) and the strong maximum principle we have

$$X \cdot \nu \equiv 0 \text{ in } L,$$

which contradicts the analyticity of the graph of v outside of the singular set S . □

Theorem 1.1 is a straightforward consequence of the Corollary above.

Using Propositions 3.2 and 3.3, we can also prove the following uniqueness result which will be used in the next section. First we set some notation, to which we will refer later.

Let $\bar{v} \geq \underline{v}$ be continuous functions on Ω with $\bar{v} = \underline{v} = \varphi$ on $\partial\Omega$, $|\bar{v}|, |\underline{v}| \leq T$. Denote by \bar{V}, \underline{V} respectively the subgraphs in C_T of the radial surfaces $X = e^{\bar{v}}z$, and $X = e^{\underline{v}}z$, $z \in \Omega$. Let

$$\mathcal{V} := \{E \subseteq C_T : E \text{ measurable, } \underline{V} \subseteq E \subseteq \bar{V}\}.$$

Lemma 3.6. *The minimization problem for $\mathcal{F}(\cdot)$ in the class \mathcal{V} admits a unique solution E . Moreover, ∂E is a radial graph over \mathbb{S}_+^n .*

Proof. Let E_1 and E_2 be distinct minimizers of \mathcal{F} in \mathcal{V} . Using (see for example [3], Lemma 15.1)

$$\mathcal{P}_{C_T}(E_1 \cap E_2) + \mathcal{P}_{C_T}(E_1 \cup E_2) \leq \mathcal{P}_{C_T}(E_1) + \mathcal{P}_{C_T}(E_2)$$

we obtain that $E_1 \cap E_2$, $E_1 \cup E_2$ also minimize \mathcal{F} in the same class. Denote by u_1, u_2 be the associated rearrangement functions (given by formula (3.4)) for these minimizers. Notice that $u_1 \neq u_2$. Indeed $E_1 \neq E_2$ implies that $E_1 \cap E_2$ has smaller volume than $E_1 \cup E_2$, and the volume is preserved by the rearrangements up to an additive constant (see (3.6)). Then according to Propositions 3.2 and 3.3, u_1 and u_2 minimize \mathcal{I}^φ in the class of all competitors v with $\underline{v} \leq v \leq \bar{v}$. Since $(u_1 + u_2)/2$ is in the same class, we can apply the same convexity argument as in Remark 2.4 to conclude that $u_1 = u_2$. Thus, we reached a contradiction. \square

4. THE DIRICHLET PROBLEM WITH SMOOTH BOUNDARY DATA.

In this section we show that upon assuming the right condition on the boundary of Ω , it is possible to set up and solve the Dirichlet problem for the energy functional $\mathcal{I}(\cdot)$ in the classical sense, that is finding a smooth minimizer v among all competitors with the same smooth boundary data. This result is of independent interest. Moreover, a corollary of this result, together with the gradient bound presented in the next section will allow us to remove the dimensional constraint of Theorem 1.1 and prove the interior smoothness of bounded BV minimizers in any dimension.

Precisely we prove the following result.

Theorem 4.1. *Let Ω be a subdomain of \mathbb{S}_+^n with $\partial\Omega \in C^2$, and let γ be a C^2 radial graph over $\partial\Omega$. Let h be the hyperbolic mean curvature of the radial cone over $\partial\Omega$ restricted to $\partial\Omega$. Then if $h > |H|$, there exists a unique smooth radial graph Σ of constant mean curvature H in \mathbb{H}^{n+1} (defined over Ω) with boundary γ .*

Theorem 1.2 follows by standard elliptic theory, combining Theorem 4.1 and the interior gradient bound Proposition 5.1 in the next section.

We first need some preliminaries. Let Σ be an hypersurface in \mathbb{H}^{n+1} and let X be the position vector of Σ in \mathbb{R}^{n+1} . We set \mathbf{n} to be a global unit normal vector field to Σ with respect to the hyperbolic metric. This determines a unit normal ν to Σ with respect to the Euclidean metric by the relation

$$\nu = \frac{\mathbf{n}}{u},$$

where u denotes the height function $u = X \cdot \mathbf{e}$. The hyperbolic principal curvatures $\kappa_1, \dots, \kappa_n$ of Σ (with respect to \mathbf{n}) are related to the Euclidean principal curvatures $\tilde{\kappa}_1, \dots, \tilde{\kappa}_n$ of Σ (with respect to ν) by the well-known formula

$$\kappa_i = u\tilde{\kappa}_i + \nu^{n+1}.$$

Therefore the hyperbolic mean curvature H and Euclidean mean curvature H_E are related by

$$(4.1) \quad H = uH_E + \nu^{n+1}.$$

Let τ_1, \dots, τ_n be a local frame of smooth vector fields on \mathbb{S}_+^n . Denote by $\sigma_{ij} = \tau_i \cdot \tau_j$ the standard metric on \mathbb{S}^n and σ^{ij} its inverse. For a function v on \mathbb{S}^n , we use the notation $v_i = \nabla_i v = \nabla_{\tau_i} v$, $v^i = \sigma^{ik} v_k$, $v_{ij} = \nabla_j \nabla_i v$, etc.

For a radial graph $X = e^v z$, the induced Euclidean metric and its inverse are given by

$$(4.2) \quad \tilde{g}_{ij} = e^{2v}(\sigma_{ij} + v_i v_j), \quad \tilde{g}^{ij} = e^{-2v} \left(\sigma^{ij} - \frac{v^i v^j}{W^2} \right),$$

where

$$(4.3) \quad W = \sqrt{1 + |\nabla v|^2}.$$

The outward unit normal to X is

$$(4.4) \quad \nu = \frac{z - \nabla v}{W},$$

and the Euclidean second fundamental form is given by

$$\tilde{b}_{ij} = \frac{e^v}{W} (v_{ij} - v_i v_j - \sigma_{ij}).$$

Therefore, using (4.2) we have

$$(4.5) \quad nH_E = \tilde{g}^{ij} \tilde{b}_{ij} = \frac{e^{-v}}{W} \left\{ \left(\sigma^{ij} - \frac{v^i v^j}{W^2} \right) v_{ij} - n \right\}.$$

Combining (4.1), (4.4), (4.5), we have

Lemma 4.2. *The radial graph $X = e^v z$ has constant hyperbolic mean curvature H if and only if v satisfies the nondivergence form elliptic equation*

$$(4.6) \quad \frac{1}{W} a^{ij} v_{ij} = \frac{n}{y} \left(H + \frac{\mathbf{e} \cdot \nabla v}{W} \right), \quad a^{ij} = \sigma^{ij} - \frac{v^i v^j}{W^2}.$$

It is easily seen that (4.6) can be written in divergence form as

$$(4.7) \quad \operatorname{div}_z \left(\frac{y^{-n} \nabla v}{\sqrt{1 + |\nabla v|^2}} \right) = nH y^{-(n+1)},$$

which is the Euler-Lagrange equation of our functional (1.3), the usual area plus nH volume functional for the hyperbolic radial graph.

Given a subdomain Ω of \mathbb{S}_+^n we can then formulate (according to Lemma 4.2) the following Dirichlet problem for a radial graph $X = e^v z$ over Ω of constant hyperbolic mean curvature H ,

$$(4.8) \quad \frac{1}{W} \left(\sigma^{ij} - \frac{v^i v^j}{W^2} \right) v_{ij} = \frac{n}{y} \left(H + \frac{\mathbf{e} \cdot \nabla v}{W} \right) \quad \text{in } \Omega,$$

$$(4.9) \quad v = \phi \quad \text{on } \partial\Omega.$$

Remark 4.3. An equivalent problem has been studied (even for prescribed mean curvature) by Nitsche [5] using a more complicated model of hyperbolic space. However as we shall see below, the problem can be easily solved directly, even for continuous boundary data.

Theorem 4.1, which is an existence and uniqueness statement for the Dirichlet problem (4.8)-(4.9), will follow from the following result, by standard elliptic theory.

Theorem 4.4. *Let h be the hyperbolic mean curvature of the radial cone C over $\partial\Omega$ restricted to $\partial\Omega$, and let $\phi \in C^2(\mathbb{S}_+^n)$. Then if $h > |H|$, there exists a unique minimizer v of $\mathcal{I}(\cdot)$ in $C^{0,1}(\Omega)$ such that $v = \phi$ continuously on $\partial\Omega$.*

The main ingredient in the proof of Theorem 4.4 is the following proposition which guarantees the existence of lower and upper barriers. The existence of such barriers can be obtained in a straightforward way using the method of [7]. We will sketch the main steps of the proof.

Proposition 4.5. *Let $\phi \in C^2(\mathbb{S}_+^n)$ and assume the solvability condition of Theorem 4.4. Then the Dirichlet problem (4.8)-(4.9) admits lower and upper barriers.*

First, we recall the definition of barriers. Let ϕ be a Lipschitz continuous function on $\partial\Omega$. For $z \in \Omega$, denote by $d(z)$ the distance of z from $\partial\Omega$ in the spherical metric.

An upper barrier \bar{v} relative to the Dirichlet problem (4.8)-(4.9) in Ω is a Lipschitz continuous function defined in a neighborhood $N_\delta = \{z \in \Omega : d(z) < \delta\}$ of $\partial\Omega$, such that \bar{v} is a supersolution in N_δ and

$$(4.10) \quad \bar{v} = \phi \text{ on } \partial\Omega; \quad \bar{v} \geq \sup_{\partial\Omega} \phi \text{ on } \partial N_\delta \cap \Omega.$$

Analogously, one can define a lower barrier \underline{v} as a subsolution in N_δ such that

$$(4.11) \quad \underline{v} = \phi \text{ on } \partial\Omega; \quad \underline{v} \leq \inf_{\partial\Omega} \phi \text{ on } \partial N_\delta \cap \Omega.$$

Remark 4.6. Let N be the interior unit normal (in the metric of the sphere) to $\partial\Omega$. Then the Euclidean mean curvature h_E of C restricted to $\partial\Omega$ is given by $h_E = \frac{n-1}{n} \mathcal{H}_{\partial\Omega}$ and so

$$(4.12) \quad h = y h_E + \mathbf{e} \cdot N = \frac{n-1}{n} y \mathcal{H}_{\partial\Omega} + \mathbf{e} \cdot N.$$

Moreover, if $\mathcal{H}_{\partial\Omega}(z)$ denotes the mean curvature at z of the parallel hypersurface at distance $d(z)$ to $\partial\Omega$ passing through z , then

$$(4.13) \quad (n-1)\mathcal{H}_{\partial\Omega}(z) = -\operatorname{div}_z \nabla d = -\Delta_z d(z).$$

We shall use these formulae in the construction of barriers in Proposition 4.5, which now follows.

Proof of Proposition 4.5. The proof follows the argument of [7] and is similar to the Euclidean case (see for example [2],[3].) For completeness, we present a sketch of the proof.

We proceed to construct an upper barrier \bar{v} . According to the definition of upper barrier and equation (4.6) we need to show that

$$(4.14) \quad M\bar{v} := \frac{1}{W}a^{ij}\bar{v}_{ij} - \frac{n}{y}\mathbf{e} \cdot \frac{\nabla \bar{v}}{W} \leq \frac{nH}{y} \text{ in } N_\delta,$$

for some δ to be chosen later. Here

$$a^{ij} = \sigma^{ij} - \frac{\bar{v}^i \bar{v}^j}{W^2},$$

and

$$W = \sqrt{1 + |\nabla \bar{v}|^2}.$$

Also we must satisfy condition (4.10). Let us pick

$$\bar{v}(z) = \phi(z) + \psi(d(z)),$$

where ψ is a C^2 function on $[0, \delta]$ satisfying

$$(4.15) \quad \psi(0) = 0, \psi'(t) \gg 1, \psi''(t) < 0,$$

$$(4.16) \quad \psi(\delta) \geq 2 \sup_{\Omega} |\phi| = M.$$

Using $|\nabla d| = 1$, $d^i d_{ij} = 0$ and $\sigma^{ij} d_{ij} = \Delta d$, (4.15) and the definition of a^{ij} , we find

$$M\bar{v} \leq \frac{\psi'}{\sqrt{1 + \psi'^2}}(\Delta d - \frac{n}{y}\mathbf{e} \cdot \nabla d) + \frac{\psi''}{(1 + \psi'^2)^{\frac{3}{2}}} + O(\frac{1}{y\sqrt{1 + \psi'^2}}).$$

Recalling Remark 4.6 we can express this as

$$(4.17) \quad M\bar{v} \leq -\frac{\psi'}{y\sqrt{1 + \psi'^2}}((n-1)y\mathcal{H}_{\partial\Omega}(z) + n\mathbf{e} \cdot N(z)) + \frac{\psi''}{(1 + \psi'^2)^{\frac{3}{2}}} + O(\frac{1}{y\sqrt{1 + \psi'^2}}).$$

Let $\psi(t) = \frac{1}{K} \log(1 + \beta t)$ where $\beta = K^2 e^{MK}$ and $\delta = K^{-2}$. Then (4.15) and (4.16) are satisfied as

$$\psi'(t) \geq \frac{\beta}{K(1+\beta\delta)} > K(1-e^{-MK}), \quad \psi'' = -K\psi'^2,$$

and

$$\psi(\delta) = \frac{1}{K} \log(1+\beta\delta) \geq M.$$

Assume the strict solvability condition $h \geq |H| + 2\epsilon_0$. Then

$$(4.18) \quad (n-1)y\mathcal{H}_{\partial\Omega}(z) + n\mathbf{e} \cdot N(z) \geq n(|H| + \epsilon_0)$$

in N_δ for small δ . Hence combining (4.17), (4.18)

$$(4.19) \quad M\bar{v} \leq -\frac{\psi'}{\sqrt{1+\psi'^2}} \frac{n(|H| + \epsilon_0)}{y} - \frac{K\psi'^2}{(1+\psi'^2)^{\frac{3}{2}}} + O\left(\frac{1}{y\sqrt{1+\psi'^2}}\right).$$

Therefore we can choose K large so that \bar{v} is an upper barrier in N_δ . Analogously $\underline{v} = \phi - \frac{1}{K} \log(1+\beta d)$ is a lower barrier in N_δ . \square

Remark 4.7. Note that when the strict solvability condition $h \geq |H| + 2\epsilon_0$ is satisfied, we obtain gradient and continuity estimates on $\partial\Omega$ that are independent of $\min_{\partial\Omega} y$.

Remark 4.8. Under certain conditions we can sharpen the solvability condition to $h \geq |H|$. Suppose $h = |H|$ at $P \in \partial\Omega$ and let

$$nh(s) = (n-1)y(z(s))\mathcal{H}_{\partial\Omega}(z(s)) + n\mathbf{e} \cdot N(z(s))$$

along the (inward) geodesic orthogonal to $\partial\Omega$ starting at P . Note that

$$\begin{aligned} \dot{y}(s) &= \mathbf{e} \cdot N(s) \\ \ddot{y}(s) &= -y(s). \end{aligned}$$

Hence from standard comparison theory (see [7])

$$(4.20) \quad \begin{aligned} n\dot{h}(s) &= (n-1)(y\dot{\mathcal{H}}_{\partial\Omega}(s) + \mathcal{H}_{\partial\Omega}(s)\mathbf{e} \cdot N(s)) - ny(s) \\ &\geq (n-1)(y(s)(\mathcal{H}_{\partial\Omega}^2(s) + 1) + \mathcal{H}_{\partial\Omega}(s)\mathbf{e} \cdot N(s)) - ny(s). \end{aligned}$$

Using $\mathbf{e} \cdot N = |H| - \frac{n-1}{n}y(P)\mathcal{H}_{\partial\Omega}$ at P in (4.20) gives

$$\begin{aligned} \frac{n}{n-1}\dot{h}(0) &\geq y(P)\mathcal{H}_{\partial\Omega}^2 + \mathcal{H}_{\partial\Omega} \left(|H| - \frac{n-1}{n}y(P)\mathcal{H}_{\partial\Omega} \right) - \frac{1}{n-1}y \\ &= \frac{y(P)}{n}\mathcal{H}_{\partial\Omega}^2 + |H|\mathcal{H}_{\partial\Omega} - \frac{1}{n-1}y. \end{aligned}$$

Hence if

$$\mathcal{H}_{\partial\Omega} > -\frac{n}{2} \left(-|H| + \sqrt{H^2 + \frac{4y^2}{n(n-1)}} \right),$$

then $\dot{h}(0) > 0$ so we obtain from (4.17) (for small δ)

$$M\bar{v} + \frac{n|H|}{y} \leq -\frac{K\psi'^2}{(1+\psi'^2)^{\frac{3}{2}}} + O\left(\frac{1}{y\sqrt{1+\psi'^2}}\right) < 0$$

if we choose K large enough (but now depending on $\min_{\partial\Omega} y$).

We now introduce some notation which we will use in the proof of Theorem 4.4.

Let K be a fixed constant, $\varepsilon > 0$, and let $\tau, |\tau| \leq 1$ be a vector lying in the hyperplane $\tau \cdot \mathbf{e} = 0$. For any bounded function w over a subdomain $\Omega \subset \mathbb{S}_+^n$ we denote by $w^* = w^*(\tau, \varepsilon)$ the corresponding possibly multivalued function such that the surface $X = e^{w+K\varepsilon}z + \tau\varepsilon$ can be represented as $X = e^{w^*}z$ over its projection Ω_w^* on the unit upper hemisphere \mathbb{S}_+^n . Precisely, let $e^{w(z)+K\varepsilon}z + \tau\varepsilon = e^{w^*(z^*)}z^*$, with $z^* \in \Omega_w^* \subset S_+^n$ and write $\rho = e^{w(z)}, \rho^* = e^{w^*(z^*)}$. Then

$$z^* = \frac{z + \varepsilon e^{-K\varepsilon} \frac{\tau}{\rho}}{\sqrt{1 + 2\varepsilon e^{-K\varepsilon} \frac{z \cdot \tau}{\rho} + \frac{e^{-2K\varepsilon} \varepsilon^2 |\tau|^2}{\rho^2}}},$$

$$\rho^* = \sqrt{e^{2K\varepsilon} \rho^2 + 2e^{K\varepsilon} \varepsilon \rho z \cdot \tau + \varepsilon^2 |\tau|^2}.$$

Note that if w is Lipschitz with constant L , then the mapping $z \rightarrow z^*$ is injective for $\varepsilon \leq \varepsilon_0(L)$ and hence w^* is well-defined and also Lipschitz. Moreover, if \bar{w} and \underline{w} are both Lipschitz with constant L and $\bar{w} = \underline{w} = \varphi$ on $\partial\Omega$, then for $\varepsilon \leq \varepsilon_0(L)$, $\Omega_{\bar{w}}^* = \Omega_{\underline{w}}^*$.

We are now ready to prove our Theorem.

Proof of Theorem 4.4. Theorem 2.3 together with Proposition 4.5 guarantees the existence (and uniqueness) of a minimizer v to \mathcal{I}^ϕ which is in the class $BV_M(\Omega) \cap C(\partial\Omega)$. We need to show that $v \in C^{0,1}(\Omega)$. Towards this aim we will prove the following claim.

Claim: For any vector $\tau, |\tau| \leq 1$, such that $\tau \cdot \mathbf{e} = 0$, and for all small $\varepsilon > 0$, the hypersurface $X = e^{v(z)+K\varepsilon}z + \tau\varepsilon$ is above the hypersurface $X = e^{v(z)}z$ in their common domain of definition.

Here K denotes a big constant depending on the Lipschitz constant of the barriers from Proposition 4.5.

First we observe that the existence of barriers implies the existence of two Lipschitz functions \underline{v}, \bar{v} such that $\underline{v} \leq v + K\varepsilon \leq \bar{v}$ (here we are using Corollary 2.5), and $\underline{v} = \bar{v} = \phi + K\varepsilon$ on $\partial\Omega$. Correspondingly, using the notation introduced before the proof, \underline{v}^* and \bar{v}^* are Lipschitz functions for small ε , and $\Omega_{\bar{v}}^* = \Omega_{\underline{v}}^* := \Omega^*$.

We wish to prove that v^* is a (single-valued) function over $\Omega_v^* = \Omega^*$. Then the desired claim consist in showing that $v^* \geq v$ in $\Omega \cap \Omega^*$, and it will follow from the comparison principle Lemma 2.6.

We use the notation at the end of Section 3. Let C be the radial cone over Ω , and set

$$\mathcal{V} + \epsilon\tau := \{E \subseteq C + \epsilon\tau : E \text{ measurable, } \underline{V} + \epsilon\tau \subseteq E \subseteq \overline{V} + \epsilon\tau\},$$

where $A + \epsilon\tau := \{x + \epsilon\tau, x \in A\}$ for all $A \subset \mathbb{R}^{n+1}$.

Also, if C^* is the radial cone over Ω^* , we let

$$\mathcal{V}^* = \{E \subseteq C^* : E \text{ measurable, } \underline{V}^* \subseteq E \subseteq \overline{V}^*\},$$

where $\underline{V}^*, \overline{V}^*$ denote respectively the subgraphs in C^* of $X = e^{\underline{v}^*}z$, and $X = e^{\overline{v}^*}z$.

Notice that there is a one-to-one correspondence between competitors in the classes $\mathcal{V} + \epsilon\tau, \mathcal{V}^*$ and the associated energies differ by a constant (recall the definition of w^*).

Hence, since the subgraph of $X = e^{v+K\epsilon}z + \epsilon\tau$ minimizes \mathcal{F} in $\mathcal{V} + \epsilon\tau$, then the subgraph of $X = e^{v^*}z$ is a minimizer to \mathcal{F} in \mathcal{V}^* , and by the uniqueness result Proposition 3.6 it is a graph over Ω^* .

Now, in order to apply the comparison principle Lemma 2.6, we need to show that

- (1) $v^* \geq v$ on $\partial\Omega^* \cap \overline{\Omega}$;
- (2) $v^* \geq v$ on $\partial\Omega \cap \overline{\Omega}^*$;

where the inequalities above are meant in the trace sense (note that the existence of barriers implies that v^* has a continuous trace on $\partial\Omega^* \cap \overline{\Omega}$, while v has a continuous trace on $\partial\Omega \cap \overline{\Omega}^*$).

In order to prove (1), we will show that v^* is greater than the upper barrier \overline{v} for v on $\partial\Omega^* \cap \overline{\Omega}$. Let $z \in \partial\Omega^* \cap \overline{\Omega}$, and let $x \in \partial\Omega$ be such that

$$e^{v^*(z)}z = e^{v(x)+K\epsilon}x + \tau\epsilon.$$

It follows that

$$|e^{v^*(z)} - e^{v(x)+K\epsilon}| \leq \epsilon$$

and

$$|x - z| \leq C\epsilon,$$

with C depending on the L^∞ norm of v . If K is very large, these two inequalities imply that

$$(4.21) \quad v^*(z) \geq v(x) + K^*|x - z|,$$

where K^* is larger than the Lipschitz constant of the upper barrier \overline{v} . Since $v(x) = \overline{v}(x)$ equation (4.21) clearly gives (1).

Part (2) follows in the same way, using the lower barrier for v^* . Thus our claim is proved.

We now show that our claim implies the Lipschitz continuity of v .

Let $z \in \Omega$ and let $\mathcal{C} = \mathcal{C}(z, \theta)$ be the circular cone with vertex at $e^{v(z)}z$, axis z , and opening θ . Since Ω is a strict subdomain of \mathbb{S}_+^n , it is above the hyperplane $y = \delta$ (recall that $y = z_{n+1}$), and thus each point x can be represented as:

$$(4.22) \quad x = e^{v(z)}z + \alpha z + \beta \sigma,$$

with $|\sigma| = 1$, $\sigma \cdot \mathbf{e} = 0$, $\alpha, \beta \geq 0$, and $\beta/\alpha \leq C(\theta, \delta)$ with $C(\theta, \delta) \rightarrow 0$ as $\theta \rightarrow 0$.

Indeed, each point x in the cone \mathcal{C} can be represented as

$$x = e^{v(z)}z + \gamma(z + \eta z_\perp)$$

with z_\perp unit vector in $T_z(\mathbb{S}_+^n)$, $\gamma \geq 0$, and $0 \leq \eta \leq \tan \theta \rightarrow 0$ as $\theta \rightarrow 0$.

Now, let us decompose

$$z_\perp = az + b\sigma$$

with

$$a = \frac{z_\perp \cdot \mathbf{e}}{z \cdot \mathbf{e}}; \quad b = \sqrt{1 + a^2}.$$

Hence $\sigma \cdot \mathbf{e} = 0$, $|\sigma| = 1$. Moreover,

$$|a| \leq 1/\delta, \quad b \leq 2/\delta$$

because z is above the hyperplane $y = \delta$.

Therefore,

$$x = e^v(z)z + \gamma[(1 + \eta a)z + b\eta\sigma]$$

with the ratio

$$\frac{b\eta}{1 + \eta a}$$

going to zero as θ goes to zero.

Now, given x (represented as in (4.22)) in a neighborhood N (in \mathcal{C}) of $e^{v(z)}z$, that is for α small, we can choose ϵ such that

$$e^{v(z)+K\epsilon} = e^{v(z)} + \alpha,$$

hence $\epsilon = O(\alpha)$. Moreover, since $\beta/\alpha \leq C(\theta, \delta) \rightarrow 0$ as $\theta \rightarrow 0$, by choosing θ small enough depending on $K, \|v\|_\infty, \epsilon_0(L), \delta$ we can guarantee that $\beta \leq \epsilon$. Hence

$$x = e^{v(z)+K\epsilon} + \epsilon\tau.$$

Thus the set $S(\epsilon, \tau) = \{X = e^{v(z)+K\epsilon}z + \epsilon\tau, 0 \leq \epsilon \leq \epsilon_0(L), |\tau| \leq 1, \tau \cdot \mathbf{e} = 0\}$ contains the cone $N \cap \mathcal{C}(z, \theta)$.

Therefore, according to our claim at each point of the surface $X = e^{v(z)}z$, there exists a small radial cone of fixed opening which is completely above the surface. This geometric property translates in the fact that for $q \in \Omega$ in a neighborhood of z we have

$$e^{v(q)} \leq e^{v(z)} + C(\theta)|z - q|.$$

Since v is bounded, this implies the Lipschitz continuity of v . \square

We state two simple corollaries of Theorem 4.1.

Corollary 4.9. *Let $B_\rho(P)$ be a ball in $\mathbb{S}_+^n \cap \{y \geq \epsilon\}$, for any $\epsilon > 0$, and let $\phi \in C^2(\mathbb{S}_+^n)$. Then there exists a constant $r_0 = r_0(n, H, \epsilon)$ such that the Dirichlet problem (4.8)-(4.9) is uniquely solvable in $C^\infty(B_\rho(P))$, for all $\rho \leq r_0$.*

Corollary 4.10. *Let S_ϵ be the spherical cap $\mathbb{S}_+^n \cap \{y > \epsilon\}$, for any $\epsilon > 0$, and let $\phi \in C^2(\mathbb{S}_+^n)$. Then the Dirichlet problem (4.8)-(4.9) is uniquely solvable in $C^\infty(S_\epsilon)$.*

5. THE INTERIOR GRADIENT BOUND AND THE PROOF OF THEOREM 1.3

5.1. The interior gradient bound. In this subsection we prove the following interior gradient bound.

Proposition 5.1. *Let v be a C^3 function satisfying equation (4.7) in $B_\rho(P) \subset \{y \geq \epsilon\}$. Then*

$$W(P) \leq C_1 e^{\frac{C_2}{\rho^2}},$$

where C_1, C_2 are non-negative constants depending only on n, H, ϵ and $\|v\|_{L^\infty}$.

Proof. Define the following linear elliptic operator

$$(5.1) \quad \mathcal{L} \equiv a^{ij} \nabla_{ij} - \frac{2}{W} a^{ij} W_i \nabla_j - \frac{n}{y} \left(H \frac{\nabla v}{W} + \mathbf{e} \right) \cdot \nabla$$

where a^{ij} and W are as in (4.6),(4.3).

Throughout the proof, the constants may depend on n, H, ϵ and $\|v\|_{L^\infty}$. One can compute that

$$(5.2) \quad \mathcal{L}W \geq -CW \quad \text{in } B_\rho(P),$$

(for details we refer the reader to Theorem 4.2 in [4], formula (4.16)).

We will derive a maximum principle for the function $h = \eta(x)W$ by computing $\mathcal{L}h$. Without loss of generality we may assume $1 \leq v \leq C_0$. A simple computation gives

$$(5.3) \quad \mathcal{L}h \geq W(M\eta - C\eta),$$

where

$$(5.4) \quad M \equiv a^{ij} \nabla_{ij} - \frac{n}{y} \left(H \frac{\nabla v}{W} + \mathbf{e} \right) \cdot \nabla.$$

Note that $Mv = \frac{nH}{yW}$. Choose

$$(5.5) \quad \eta(z) \equiv g(\phi(z)); \quad g(\phi) = e^{K\phi} - 1,$$

with the constant $K > 0$ to be determined and

$$\phi(z) = \left[-\frac{v(z)}{2v(P)} + \left(1 - \left(\frac{d_P(z)}{\rho} \right)^2 \right) \right]^+.$$

Here $d_P(z)$ is the distance function (on the sphere) from P , the center of the geodesic ball $B_\rho(P)$.

Since v is positive, $\eta(z)$ has compact support in $B_\rho(P)$. We will choose K so that $M\eta > C\eta$ on the set where $h > 0$ and W is large (here M is as in (5.4)).

A straightforward computation gives that on the set where $h > 0$,

$$\begin{aligned} M\eta &= g'(\phi) \left(a^{ij} \nabla_{ij} \phi - \frac{n}{y} \left(H \frac{\nabla v}{W} + \mathbf{e} \right) \cdot \nabla \phi \right) + g''(\phi) a^{ij} \nabla_i \phi \nabla_j \phi \\ &= K e^{K\phi} \left\{ -\frac{1}{2v(P)} \frac{nH}{yW} - \frac{2}{\rho^2} (d_P a^{ij} \nabla_{ij} d_P + a^{ij} \nabla_i d_P \nabla_j d_P) \right. \\ &\quad \left. - \frac{n}{\rho^2 y} \left(H \frac{\nabla v}{W} + \mathbf{e} \right) \cdot d_P \nabla d_P \right\} \\ &\quad + K^2 e^{K\phi} a^{ij} \left(\frac{v_i}{2v(P)} + \frac{2}{\rho^2} d_P \nabla_i d_P \right) \left(\frac{v_j}{2v(P)} + \frac{2}{\rho^2} d_P \nabla_j d_P \right). \end{aligned}$$

Using the definition of a^{ij} we find ($\langle \cdot, \cdot \rangle$ denotes the inner product with respect to the induced Euclidean metric on Σ)

$$\begin{aligned} a^{ij} \left(\frac{v_i}{2v(P)} + \frac{2}{\rho^2} d_P \nabla_i d_P \right) \left(\frac{v_j}{2v(P)} + \frac{2}{\rho^2} d_P \nabla_j d_P \right) &= \\ \frac{|\nabla v|^2}{4v(P)^2} + \frac{2d_P}{\rho^2 v(P)} \langle \nabla v, \nabla d_P \rangle + \frac{4d_P^2}{\rho^4} \left(1 - \left(\left\langle \frac{\nabla v}{W}, \nabla d_P \right\rangle \right)^2 \right). \end{aligned}$$

Hence,

$$M\eta - C\eta \geq e^{K\phi} \left\{ K^2 \left(\frac{1}{8C_0^2} - \frac{1}{W^2} \left(\frac{1}{\rho^2} + \frac{1}{8C_0^2} \right) \right) - CK \frac{1}{\rho^2} - C \right\}.$$

Therefore on the set where $h > 0$ and $W > 1 + 4\frac{C_0}{\rho}$ we find

$$M\eta - C\eta \geq e^{K\phi} \left\{ \frac{K^2}{16C_0^2} - CK \frac{1}{\rho^2} - C \right\}.$$

Thus, the choice $K = 16CC_0 \left(1 + \frac{C_0}{\rho^2} \right)$ gives

$$M\eta - C\eta \geq 15C e^{K\phi} > 0$$

on the set where $h > 0$ and $W > 1 + 4\frac{C_0}{\rho}$. Hence by (5.3) and the maximum principle, $W \leq 1 + 4\frac{C_0}{\rho}$ at the point Q where h achieves its maximum. Therefore

$$h(P) = (e^{\frac{K}{2}} - 1)W(P) \leq h(Q) \leq (1 + 4\frac{C_0}{\rho})(e^K - 1),$$

and hence

$$(5.6) \quad W(P) \leq e^{\frac{CC_0^2}{\rho^2}}$$

for a slightly larger constant C . This proves Proposition 5.1. \square

5.2. Smoothness of minimizers in any dimension. In this subsection we remove the dimensional constraint and prove the regularity result in Theorem 1.3. The proof follows the lines of the Euclidean case. We present it for the sake of completeness.

Proof of Theorem 1.3. We use a standard approximation argument. Let $B = B_\rho(P)$ be a ball in Ω , with $\rho \leq r_0$ and r_0 as in Corollary 4.9.

Denote by $\tilde{S} := \text{proj}_\Omega S$. Since \tilde{S} satisfies $H_{n-6}(\tilde{S}) = 0$, there exists a sequence S_k of open sets, such that

$$S_k \supset \supset S_{k+1}, \quad k = 1, 2, 3, \dots \quad \bigcap_{k \in \mathbb{N}} S_k = \tilde{S}$$

and also

$$H_{n-1}(S_k \cap \partial B) \rightarrow 0.$$

Now let ϕ_k be a smooth function on ∂B satisfying

$$\phi_k = v \text{ in } \partial B \setminus S_k$$

$$(5.7) \quad \sup_{\partial B} |\phi_k| \leq 2 \sup_{\partial B} |v|.$$

Let v_k be the unique solution to the Dirichlet problem with boundary data ϕ_k on ∂B (see Corollary 4.9). The functions v_k 's are smooth in B and also according to (5.7) and Theorem 2.3

$$(5.8) \quad \sup_B |v_k| \leq M(\sup_{\partial B} |v|).$$

We also have that the v_k minimizes $\mathcal{I}_B(\cdot)$ among all competitors with boundary data ϕ_k (see Remark 2.7). Hence,

$$(5.9) \quad \mathcal{I}_B(v_k) \leq \mathcal{I}_B^{\phi_k}(w)$$

for every $w \in BV(B)$. In particular, for $w = 0$,

$$(5.10) \quad \mathcal{I}_B(v_k) \leq |B| + \int_{\partial B} |\phi_k| dH_{n-1} \leq C$$

where in the last inequality we used (5.7).

From (5.8) and the a priori estimate of the gradient (Proposition 5.1) we conclude that the gradients ∇v_k are equibounded in every compact subset of B . Hence, by Ascoli-Arzelà we can extract a subsequence, which we still denote by v_k , which converges uniformly on compact subsets of B to a Lipschitz continuous function \tilde{v} . Moreover, by the lower semicontinuity of $\mathcal{I}_B(\cdot)$ combined with (5.8) and (5.10) we obtain

$$\int_B |\nabla \tilde{v}| \leq C$$

and therefore $\tilde{v} \in W^{1,1}(B)$.

We claim that \tilde{v} has trace v on ∂B . Assuming that the claim is true, then passing to the limit in (5.9) with $w = v$ and remarking that $\phi_k \rightarrow v$ in $L^1(\partial B)$ we have

$$\mathcal{I}_B(\tilde{v}) \leq \mathcal{I}_B(v).$$

Thus the function \tilde{v} also minimizes $\mathcal{I}_B(\cdot)$ and by the uniqueness of minimizers (see Remark 2.4) we obtain $v = \tilde{v}$ proving that v is Lipschitz continuous in B . Hence, by elliptic regularity v is analytic in B .

We are now left with the proof of the claim. Let $z_0 \in \partial B$ be a regular point for v . Then for k large enough $z_0 \in \partial B \setminus S_k$ and hence $\phi_j = v$ in a neighborhood of z_0 in ∂B , for all $j \geq k$. We can construct two C^2 functions $\underline{\phi}$ and $\overline{\phi}$ on ∂B , such that $\underline{\phi} = \overline{\phi} = v$ in a neighborhood of z_0 and $\underline{\phi} \leq \phi_j \leq \overline{\phi}$ for all $j \geq k$.

Now, we solve the Dirichlet problem with boundary data $\underline{\phi}, \overline{\phi}$ and denote the solutions respectively by $\underline{v}, \overline{v}$ (again we use Corollary 4.9). Then, $\underline{v} \leq v_j \leq \overline{v}$ for all $j \geq k$ and therefore $\underline{v} \leq \tilde{v} \leq \overline{v}$, which immediately yields $\tilde{v}(z_0) = v(z_0)$.

Thus, $\tilde{v} = v$ at every regular point, which implies the desired claim since $H_{n-1}(\tilde{S}) = 0$. \square

We conclude this section by sketching the proof of Theorem 1.4.

Proof of Theorem 1.4. Assume that Γ is represented by

$$X = e^\varphi z, z \in \partial \mathbb{S}_+^n,$$

with $\varphi \in C^2(\overline{\mathbb{S}_+^n})$. Then, according to Proposition 4.5 (see Remark 4.7) we can find upper and lower barriers \overline{v} and \underline{v} coinciding with φ on $\partial \mathbb{S}_+^n$. For any small $\epsilon > 0$, let ψ_ϵ be a smooth function on the spherical cap $S_\epsilon := \mathbb{S}_+^n \cap \{y > \epsilon\}$ such that $\underline{v} \leq \psi_\epsilon \leq \overline{v}$ on the boundary of ∂S_ϵ . Let v_ϵ be a minimizer to $\mathcal{I}_{S_\epsilon}^{\psi_\epsilon}(\cdot)$, which by our regularity theory is smooth. By the comparison principle (Corollary 2.5)

$\underline{v} \leq v_\epsilon \leq \bar{v}$ in S_ϵ . By the interior a priori bound (Proposition 5.1) we can extract a subsequence v_{ϵ_k} which converges uniformly on compacts of \mathbb{S}_+^n to a function v which solves the equation and also $\underline{v} \leq v \leq \bar{v}$ in \mathbb{S}_+^n . This implies the continuity of v up to the boundary.

Finally, if φ is only continuous, we approximate it (from above and below) with C^2 functions, and conclude the argument by comparison with the barriers associated to the smooth approximated boundary data.

□

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